

# RANDOM COLOURINGS OF APERIODIC GRAPHS: ERGODIC AND SPECTRAL PROPERTIES

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**ABSTRACT.** We study randomly coloured graphs embedded into Euclidean space, whose vertex sets are infinite, uniformly discrete subsets of finite local complexity. We construct the appropriate ergodic dynamical systems, explicitly characterise ergodic measures, and prove an ergodic theorem. For covariant operators of finite range defined on those graphs, we show the existence and self-averaging of the integrated density of states, as well as the non-randomness of the spectrum. Our main result establishes Lifshits tails at the lower spectral edge of the graph Laplacian on bond percolation subgraphs, for sufficiently small probabilities. Among other assumptions, its proof requires exponential decay of the cluster-size distribution for percolation on rather general graphs.

## 1. INTRODUCTION

Studying ensembles of random graphs is a broad subject with many different facets. One of them, spectral properties of random graphs, has found increasing interest in recent years. Its goal is to determine spectral properties of the graph Laplacian, or of similar operators associated with the graph, and to investigate their relation to the graph structure. Erdős-Rényi random graphs constitute one class of examples, for which such types of results are known by now [KhSV, KhKM, BDJ].

Another class of random graphs consists of those generated by a percolation process from an underlying graph (“base graph”), which is embedded into  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ . Standard Bernoulli (bond- or site-) percolation subgraphs of the  $d$ -dimensional hypercubic lattice are the prime example in this category [G]. Here, ergodicity with respect to translations has fundamental consequences, such as non-randomness of the spectrum, as well as existence and self-averaging of the integrated density of states [V, KiM]. The behaviour of the integrated density of states near the edges of the spectrum requires a more detailed understanding. Lifshits-tail behaviour was found in the non-percolating phase [KiM], while the percolating cluster may give rise to a van Hove asymptotics [MS]. In this context, techniques from the theory of random Schrödinger operators have turned out to be very efficient. Furthermore, the connection to the theory of random walks in random environments [Hu, Bar] was exploited. Very recently, results of [KiM, MS] have been extended to amenable Cayley graphs [AV1, AV3]. There, it is invariance under the appropriate group action, which replaces translational invariance.

But how important is the automorphism group of the base graph for the spectral asymptotics of its percolation subgraphs? To pursue this question, we consider base graphs whose vertex sets are given by infinite, uniformly discrete subsets of  $\mathbb{R}^d$ , with the property of finite local complexity (see Definition 2.1 (i) below). Examples include quasiperiodic tilings such as a Penrose tiling (see [BaaM] for a recent monograph on quasiperiodic point sets), more generally, tilings with a finite set of prototiles [GS], but also random tiling ensembles [RHHB]. Typically, none of these enjoys invariance under an appropriate group action. Ergodic and spectral properties of the base graphs were first derived by [Ho1, Ho2], and significantly extended by [LS1, KILS, LS2], using methods from dynamical systems. In this paper, we supply these base graphs with a random colouring and study their spectral properties. The main result of this paper, Theorem 5.2, goes beyond basic ergodic spectral properties and establishes Lifshits-tail behaviour at the lower spectral edge for the graph Laplacian on percolation subgraphs.

Our proof of this result involves three preparatory steps, each of which is interesting in its own. The first step belongs to the realm of dynamical systems theory, the second to spectral theory, and the third to percolation theory.

(i) *Construct the appropriate ergodic dynamical systems, explicitly characterise ergodic measures and prove an ergodic theorem.* Given an ergodic measure on the dynamical system of the base graphs, we will explicitly construct an ergodic measure for corresponding randomly coloured graphs, following ideas of [Ho3]. The main result of this step is an ergodic theorem (Theorem 3.5) for dynamical systems associated with randomly coloured graphs. It extends [Ho3], where colourings of aperiodic Delone graphs with strictly ergodic dynamical system have been studied. Our setting covers the full range from periodic structures to random tilings. Moreover, we do not require relative denseness of the vertex sets, thereby including examples such as the visible lattice points [BaaMP] in our setup. Apparently, some of the technical problems we had to overcome are closely related to ones in [BaaZ], where diffraction properties of certain random point sets, including percolation subsets, have been investigated very recently.

(ii) *Derive ergodic spectral properties of covariant, finite-range operators on randomly coloured aperiodic graphs.* Theorem 4.5 characterises the integrated density of states of such an operator by a macroscopic limit. Theorem 4.7 states the non-randomness of the spectrum of the operator and relates it to the set of growth points of the integrated density of states. In particular, the theorems guarantee that there are no exceptional instances to their statements for uniquely ergodic systems. We provide elementary proofs of Theorems 4.5 and 4.7. In the absence of a colouring, corresponding results have been derived in [Ho1, Ho2, LS1, LS2], mainly in the strictly ergodic or in the uniquely ergodic case.

(iii) *Establish exponential decay of the cluster-size distribution in the non-percolating phase for general graphs.* We derive an elementary exponential-decay estimate for the probability to find an open path from the centre to the complement of a large ball. Unfortunately, this estimate holds only for sufficiently small bond probabilities. For these probabilities,

the decay of the cluster-size distribution then follows from that estimate, by verifying that the corresponding arguments in [G] apply also in our general setting. Exponential decay throughout the non-percolating phase for quasi-transitive graphs has been proved recently [AV2]. Within our more general setup, an extension to higher bond probabilities up to criticality remains a challenging open question, see also the discussion in [Ho3].

The manuscript [LV], which was finalised at the same time as ours, establishes uniform convergence in the energy of the finite-volume approximants to the integrated density of states under rather general conditions. In particular, it applies to percolation on Delone dynamical systems and thus improves on Theorem 4.5 under slightly different conditions. However, the validity of our general Ergodic Theorem 3.5 is an open question in the approach of [LV]. Using uniform convergence would not allow to strengthen our main result on Lifshits tails in Theorem 5.2.

Our paper is organised as follows. Section 2 sets the notation and introduces dynamical systems associated with uncoloured graphs. This is a slight extension of the setup for Delone dynamical systems, such as in [LeMS]. In Section 3, we construct the dynamical systems for the corresponding randomly coloured graphs and deal with step (i). Section 4 introduces covariant operators of finite range on randomly coloured graphs and treats step (ii). Section 5 is devoted to our main result on Lifshits tails together with its proof. Section 6 contains the proof of Theorem 4.7, and Section 7 deals with step (iii).

## 2. DYNAMICAL SYSTEMS FOR GRAPHS

For the basic notions involving graphs, we refer, for example, to the textbook [D]. We consider (simple) graphs  $G = (\mathcal{V}, \mathcal{E})$ , whose vertex sets  $\mathcal{V} \equiv \mathcal{V}_G$  are countable subsets of  $\mathbb{R}^d$ . We say that  $\mathcal{V}$  is *uniformly discrete* of radius  $r \in ]0, \infty[$ , if any open ball of radius  $r$  in  $\mathbb{R}^d$  contains at most one element of  $\mathcal{V}$ . The vertex set is called *relatively dense* if there exists  $R \in [0, \infty[$  such that every closed ball of radius  $R$  contains at least one vertex. The vertex set is called a *Delone set*, if it is both uniformly discrete and relatively dense. The edge set  $\mathcal{E} \equiv \mathcal{E}_G$  of  $G$  is a subset of the set of all unordered pairs of vertices. We denote an edge by  $e \equiv \{v, w\}$ , where  $v, w \in \mathcal{V}$  with  $v \neq w$ . In other words, we do not allow self-loops, nor multiple edges between the same pair of vertices.

Recall that a graph  $G' = (\mathcal{V}', \mathcal{E}')$  is called a subgraph of  $G$ , in symbols  $G' \subseteq G$ , if  $\mathcal{V}' \subseteq \mathcal{V}$  and  $\mathcal{E}' \subseteq \mathcal{E}$ . For  $x \in \mathbb{R}^d$ , the *translated graph*  $x + G$  has vertex set  $x + \mathcal{V} := \{x + v : v \in \mathcal{V}\}$  and edge set  $x + \mathcal{E} := \{\{x + v, x + w\} : \{v, w\} \in \mathcal{E}\}$ . Given any Borel set  $B \subseteq \mathbb{R}^d$ , the *restriction*  $G \wedge B$  of  $G$  to  $B$  is the induced subgraph of  $G$  with vertex set  $\mathcal{V} \cap B$ , that is,  $\{u, v\}$  belongs to the edge set of  $G \wedge B$ , if and only if  $\{u, v\} \in \mathcal{E}$  and  $u, v \in \mathcal{V} \cap B$ . If  $B$  is bounded, then  $G \wedge B$  is called a *B-pattern* (or simply a pattern) of  $G$ . Two patterns  $P, Q$  are called *equivalent*, if  $x + P = Q$  for some  $x \in \mathbb{R}^d$ . An *r-pattern* is a pattern  $G \wedge B_r(v)$  for some  $v \in \mathcal{V}_G$ . Here we have used

the notation  $B_r(x)$  for the open ball of radius  $r > 0$  around  $x \in \mathbb{R}^d$ . In particular, we set  $B_r := B_r(0)$ . We write  $|M|$  for the cardinality of a set  $M$ .

**Definition 2.1.** (i) A set  $\mathcal{G}$  of graphs is said to have *finite local complexity*, if for every  $r > 0$

$$|\{(-v + G) \wedge B_r : v \in \mathcal{V}_G, G \in \mathcal{G}\}| < \infty. \quad (2.1)$$

In particular, a single graph  $G$  has finite local complexity if, for any given  $r > 0$ , the number of its non-equivalent  $r$ -patterns is finite.

(ii) Let  $G$  be a finite graph and  $P \subseteq G$  a pattern of  $G$ . The *number of occurrences*

$$\nu(P|G) := |\{x \in \mathbb{R}^d : x + P \subseteq G\}| \quad (2.2)$$

of  $P$  in  $G$  is the (finite) number of translates of  $P$  in  $G$ .

Geometric properties of some set of graphs  $\mathcal{G}$  are reflected by properties of an associated dynamical system. This we introduce along the lines of [LeMS], where the case of Delone multi-sets was considered. The statements of this section are proved by slight adaptations of the arguments laid down in [LeMS, RW, S]. In fact, examples of our setup include the Delone multi-sets of [LeMS], in which case  $\mathcal{G}$  is finite, vertex sets are Delone sets and edge sets are empty.

For simplicity, let us assume now that the vertex set of each  $G \in \mathcal{G}$  is uniformly discrete. Following [LeMS, Ho2], we define a *metric* on  $\mathcal{G}$  by setting

$$\text{dist}(G, G') := \min \left\{ 2^{-1/2}, \inf \{ \varepsilon > 0 : \text{there exists } x, y \in B_\varepsilon : \right. \\ \left. (x + G) \wedge B_{1/\varepsilon} = (y + G') \wedge B_{1/\varepsilon} \} \right\} \quad (2.3)$$

for all  $G, G' \in \mathcal{G}$ . In essence, two graphs are close, if they agree, up to a small translation, on a large ball around the origin. Symmetry and the triangle inequality of the metric are seen to hold for any set of graphs  $\mathcal{G}$ . The uniform discreteness assumption ensures positive definiteness, but it is much stronger than what is required. In fact, it would have been sufficient to assume merely closedness of the vertex sets and of the edge sets (with respect to a suitable metric on  $\mathcal{E}$ ). Now we define the complete metric space

$$X_{\mathcal{G}} := \overline{\{x + G : x \in \mathbb{R}^d, G \in \mathcal{G}\}}, \quad (2.4)$$

of all translates of graphs in  $\mathcal{G}$ , where the metric (2.3) is used for completion. Later we will need to know that certain properties of graphs in  $\mathcal{G}$  do not get lost in the closure.

**Lemma 2.2.** *Let  $\mathcal{G}$  be a set of graphs with uniformly discrete vertex sets.*

(i) *If for some  $d_{\max} \in \mathbb{N}$  the estimate*

$$\sup_{v \in \mathcal{V}_G} d_G(v) \leq d_{\max} \quad (2.5)$$

*holds for all  $G \in \mathcal{G}$ , then it holds also for all  $G \in X_{\mathcal{G}}$ .*

(ii) *In addition to uniform discreteness, assume there is some finite  $R > 0$ , such that the vertex sets of all  $G \in \mathcal{G}$  are relatively dense with radius  $R$ , and that*

$$\ell_{\max} := \sup\{|u - v| : G \in \mathcal{G}, \{u, v\} \in \mathcal{E}_G\} < \infty, \quad (2.6)$$

*i.e., there exists a finite maximum bond length. Then, every  $G \in X_{\mathcal{G}}$  is infinite, and if no  $G \in \mathcal{G}$  possesses a finite cluster, then no  $G \in X_{\mathcal{G}}$  possesses a finite cluster.*

*Proof.* By contradiction.

(i) Assume there exists  $G \in X_{\mathcal{G}}$  and  $v \in \mathcal{V}_G$  such that  $d_G(v) > d_{\max}$ . Choose  $0 < \varepsilon < 1/3$  small enough, such that all neighbours of  $v$  lie in the ball  $B_{\varepsilon^{-1}-1}(v)$ . Hence we have  $d_G(v) = d_{G \wedge B_{1/\varepsilon}(v)}(v)$ . Since  $G$  is in the closure of  $\mathcal{G}$ , there exists  $G' \in \mathcal{G}$  with  $\text{dist}(G, G') < \varepsilon$ . Altogether, this implies  $d_{G' \wedge B_{1/\varepsilon}(v)}(v) > d_{\max}$ , a contradiction.

(ii) The assumption of relative denseness implies that  $X_{\mathcal{G}}$  contains only infinite graphs. So let us assume there exists  $G \in X_{\mathcal{G}}$  with a finite cluster  $C$ . Let  $r_C \in ]0, \infty[$  big enough such that  $\mathcal{V}_C \subset B_{r_C}$ , and choose  $\varepsilon > 0$  so small that  $1/\varepsilon > r_C + \ell_{\max} + 3$ . Since  $G$  is in the closure of  $\mathcal{G}$ , there exists  $G' \in \mathcal{G}$  with  $\text{dist}(G, G') < \varepsilon$ , and  $G' \wedge B_{1/\varepsilon}$  has a finite cluster that cannot merge with other clusters when removing the restriction to the ball  $B_{1/\varepsilon}$ .  $\square$

Standard arguments show that the translation group  $\mathbb{R}^d$  acts continuously on  $X_{\mathcal{G}}$ , that is, the map  $G \mapsto x + G$  is continuous for every  $x \in \mathbb{R}^d$ . Thus, the triple  $(X_{\mathcal{G}}, \mathbb{R}^d, +)$  constitutes a topological dynamical system. The following result can be proved along the lines of [RW, S].

**Lemma 2.3.** *Let  $\mathcal{G}$  be set of graphs with uniformly discrete vertex sets. Then,  $X_{\mathcal{G}}$  is compact if and only if  $\mathcal{G}$  has finite local complexity.*  $\square$

As above, uniform discreteness can be replaced by closedness of all vertex and edge sets, without jeopardising the validity of Lemma 2.3. But from now on, we assume that uniform discreteness holds even uniformly in  $\mathcal{G}$ , that is, there exists  $r > 0$  such that  $\mathcal{V}_G$  is uniformly discrete of radius  $r$ , for all  $G \in \mathcal{G}$ .

Compactness of  $X_{\mathcal{G}}$  implies the existence of ergodic probability measures on the Borel-sigma algebra of  $X_{\mathcal{G}}$ , in other words  $X_{\mathcal{G}}$  is *ergodic* (w.r.t. translations). Recall that a topological dynamical system is called *uniquely ergodic*, if it carries exactly one ergodic measure. Ergodic theorems for a compact dynamical system with  $\mathbb{R}^d$ -action are given in [LeMS, Thms. 4.2 and 2.6], see also [LS2, Thm. 1] for a stronger statement in the case of minimal ergodic systems. We quote a version patterned after [LeMS] and introduce *cylinder sets*

$$\Xi_{P,U} := \{G \in X_{\mathcal{G}} : x + P \subseteq G \text{ for some } x \in U\} \subset X_{\mathcal{G}}. \quad (2.7)$$

Here,  $P$  is a pattern of some graph  $G \in \mathcal{G}$ , and  $U \subseteq \mathbb{R}^d$  is a Borel set. We write  $\text{vol}(U)$  for its Lebesgue measure.

**Theorem 2.4.** *Let  $\mathcal{G}$  be a set of graphs of finite local complexity and with uniformly discrete vertex sets of radius  $r > 0$ . Fix an ergodic probability measure  $\mu$  on  $X_{\mathcal{G}}$ . Then, given any function  $\phi \in L^1(X_{\mathcal{G}}, \mu)$ , the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \int_{B_n} dx \phi(x + G) = \int_{X_{\mathcal{G}}} d\mu(F) \phi(F) \quad (2.8)$$

*exist for  $\mu$ -a.a.  $G \in X_{\mathcal{G}}$ . If  $X_{\mathcal{G}}$  is even uniquely ergodic and, in addition, if  $\phi$  is either continuous or a linear combination of indicator functions of cylinder sets, then the limit (2.8) exists for all  $G \in X_{\mathcal{G}}$ .  $\square$*

The ergodic theorem can be used to analyse the vertex density of graphs and the asymptotic number of occurrences of patterns in graphs.

**Corollary 2.5.** *Let  $\mathcal{G}$  be a set of graphs of finite local complexity and with uniformly discrete vertex sets of radius  $r > 0$ . Fix an ergodic probability measure  $\mu$  on  $X_{\mathcal{G}}$ .*

(i) *Then there is a Borel set  $\overline{X} \subseteq X_{\mathcal{G}}$  of full  $\mu$ -measure,  $\mu(\overline{X}) = 1$ , such that the vertex density*

$$\varrho := \lim_{n \rightarrow \infty} \frac{|\mathcal{V}_G \cap B_n|}{\text{vol}(B_n)} = \int_{X_{\mathcal{G}}} d\mu(F) \sum_{v \in \mathcal{V}_F} \psi(v) \quad (2.9)$$

*exists for all  $G \in \overline{X}$ . Here,  $\psi : \mathbb{R}^d \rightarrow \mathbb{R}_{\geq 0}$  is any continuous function with support in  $B_r$  and  $\int_{\mathbb{R}^d} dx \psi(x) = 1$  (“mollifier”). In particular,  $\varrho \in [0, 1/\text{vol}(B_r)]$  is independent of  $G \in \overline{X}$  and of the choice of the mollifier. If  $X_{\mathcal{G}}$  is even uniquely ergodic, then the above statements hold with  $\overline{X} = X_{\mathcal{G}}$ .*

(ii) *The statements of Part (i) apply also to the density of vertices belonging to infinite components*

$$\varrho_{\infty} := \lim_{n \rightarrow \infty} \frac{|\mathcal{V}_{G,\infty} \cap B_n|}{\text{vol}(B_n)} = \int_{X_{\mathcal{G}}} d\mu(F) \sum_{v \in \mathcal{V}_{F,\infty}} \psi(v). \quad (2.10)$$

*Here,  $\mathcal{V}_{G,\infty} := \{v \in \mathcal{V}_G : |C_v| = \infty\}$ , with  $C_v$  denoting the cluster of  $G$  which  $v \in \mathcal{V}_G$  belongs to.*

(iii) *Let  $P$  be a pattern of some graph  $G \in \mathcal{G}$ . Then there is a Borel set  $\overline{X} \subseteq X_{\mathcal{G}}$  of full  $\mu$ -measure, such that the pattern frequency*

$$\nu(P) := \lim_{n \rightarrow \infty} \frac{\nu(P|G \wedge B_n)}{\text{vol}(B_n)} \quad (2.11)$$

*exists for all  $G \in \overline{X}$  and is independent of  $G$ . The dynamical system  $X_{\mathcal{G}}$  is uniquely ergodic, if and only if, given any pattern  $P$  of a graph in  $\mathcal{G}$ , the limit*

$$\nu(P) := \lim_{n \rightarrow \infty} \frac{\nu(P|G \wedge B_n(a))}{\text{vol}(B_n)} \quad (2.12)$$

*exists uniformly in  $G \in X_{\mathcal{G}}$  and in  $a \in \mathbb{R}^d$ , and is independent of  $G$  and  $a$ .*

(iv) *Let  $P$  be a pattern of some graph  $G \in \mathcal{G}$ , and let  $U \subset \mathbb{R}^d$  be a Borel set with diameter  $\text{diam}(U) < r$ . Then, the probability of the associated cylinder set (2.7) is given by*

$$\mu(\Xi_{P,U}) = \text{vol}(U) \nu(P). \quad (2.13)$$

**Remark 2.6.** (i) The sum over  $v$  in the  $\mu$ -integral in (2.9) contains at most one term, because the mollifier  $\psi$  is supported in the ball  $B_r$ , where  $r$  is the radius of uniform discreteness of the graphs.

(ii) The criterion (2.12) for unique ergodicity in Lemma 2.5 (iii) is often referred to as *uniform pattern frequencies*.

(iii) For later reference we give two different conditions that imply  $\varrho_\infty > 0$ : (1) The situation described in Lemma 2.2 (ii), assuming that no  $G \in \mathcal{G}$  possesses a finite cluster. (2) The dynamical system  $X_{\mathcal{G}}$  is uniquely ergodic and there exists  $G \in X_{\mathcal{G}}$  such that  $|\mathcal{V}_{G,\infty} \cap B_n|/\text{vol}(B_n)$  has a strictly positive limit as  $n \rightarrow \infty$ .

*Proof of Corollary 2.5.* Part (i) of the corollary follows from an application of Theorem 2.4 to the continuous function  $\phi(G) := \sum_{v \in \mathcal{V}_G} \psi(v)$  and the relation

$$|\mathcal{V}_G \cap B_n| = \int_{B_n} dx \phi(x + G) + \mathcal{O}(n^{d-1}). \quad (2.14)$$

The latter reveals the independence of the right-hand side of (2.9) on the particular choice of the mollifier  $\psi$ . For Part (ii), one has to replace  $\mathcal{V}_G$  by  $\mathcal{V}_{G,\infty}$  in the argument.

Parts (iii) and (iv) follow from repeating the arguments in [LeMS, Lemma 4.3 and Thm. 2.7], where the case of Delone multi-sets was treated.  $\square$

**Definition 2.7.** Let  $\mathcal{G}$  be a set of graphs of finite local complexity and with uniformly discrete vertex sets of radius  $r > 0$ . We say that the dynamical system  $X_{\mathcal{G}}$  satisfies the *positive lower frequency condition*, if for every  $G \in X_{\mathcal{G}}$  and every pattern  $P \subset G$

$$\liminf_{n \rightarrow \infty} \frac{\nu(P|G \wedge B_n)}{\text{vol}(B_n)} > 0. \quad (2.15)$$

Loosely spoken, any pattern  $P$  that occurs once in  $G$ , does so sufficiently often.

**Remarks 2.8.** (i) Minimality of  $X_{\mathcal{G}}$ , which is equivalent to repetitivity for Delone systems of finite local complexity [LaP, Thm. 3.2], implies that the positive lower frequency condition holds.

(ii) If, in addition to the positive lower frequency condition, one assumes that  $X_{\mathcal{G}}$  is uniquely ergodic, then, in view of Corollary 2.5 (iii), the  $\liminf$  in (2.15) equals the pattern frequency  $\nu(P)$ . Moreover, in this case the system is minimal and thus strictly ergodic, compare [LaP].

### 3. ERGODIC PROPERTIES OF RANDOMLY COLOURED GRAPHS

In this section, we supply the graphs of the previous section with a random colouring, and derive a corresponding extension of the Ergodic Theorem 2.4.

We fix a finite, nonempty set  $\mathbb{A}$ , equipped with the discrete topology, which we call the set of available *colours*. For definiteness, we consider only random edge colourings of graphs. But all results of this and the next section

remain valid in the case of a random colouring of vertices, and of a random colouring of both edges and vertices. This is merely a matter of notation.

For a given a graph  $G$ , we define the probability space  $\Omega_G := \times_{e \in \mathcal{E}_G} \mathbb{A}$ , equipped with the  $|\mathcal{E}_G|$ -fold product sigma-algebra  $\bigotimes_{e \in \mathcal{E}_G} 2^{\mathbb{A}}$  of the power set of  $\mathbb{A}$  and the product probability measure  $\mathbb{P}_G := \bigotimes_{e \in \mathcal{E}_G} \mathbb{P}_0$ . Here,  $\mathbb{P}_0$  is some fixed probability measure on  $\mathbb{A}$ . In other words, colours are distributed identically and independently to all edges, and the elementary event  $\omega \equiv (\omega_e)_{e \in \mathcal{E}_G} \in \Omega_G$  specifies a particular realisation of colours assigned to the edges of  $G$ .

At first we are going to extend the framework of the previous section to coloured graphs. Given a graph  $G$  and  $\omega \in \Omega_G$ , the pair  $G^{(\omega)} \equiv (G, \omega)$  is called a *coloured graph*. For any Borel set  $B \subseteq \mathbb{R}^d$ , we define the restriction  $\omega \wedge B \in \times_{e \in \mathcal{E}_{G \wedge B}} \mathbb{A}$  as the image of  $\omega$  under the canonical projection from  $\Omega_G$  to  $\times_{e \in \mathcal{E}_{G \wedge B}} \mathbb{A}$ . Likewise, given any  $x \in \mathbb{R}^d$ , the translated colour realisation  $x + \omega \in \Omega_{x+G}$  is defined component-wise by  $(x + \omega)_{x+e} := \omega_e$  for all  $e \in \mathcal{E}_G$ . We define the translation and restriction of a coloured graph in the natural way

$$x + G^{(\omega)} := (x + G)^{(x+\omega)}, \quad G^{(\omega)} \wedge B := (G \wedge B)^{(\omega \wedge B)}, \quad (3.1)$$

by shifting and truncating  $\omega$  along with  $G$ . We write  $P^{(\eta)} \subseteq G^{(\omega)}$ , if  $P \subseteq G$  and  $\eta_e = \omega_e$  for all edges  $e \in \mathcal{E}_P$ . The notions of a pattern of a coloured graph and of the number of occurrences of a finite coloured graph  $P^{(\eta)}$  in  $G^{(\omega)}$  translate accordingly from those in the previous section.

For a given set of graphs  $\mathcal{G}$ , we consider the induced set of coloured graphs

$$\widehat{\mathcal{G}} := \{G^{(\omega)} : \omega \in \Omega_G, G \in \mathcal{G}\}. \quad (3.2)$$

**Remark 3.1.** Since  $\mathbb{A}$  provides only finitely many different colours, it follows that  $\mathcal{G}$  has finite local complexity, if and only if  $\widehat{\mathcal{G}}$  has finite local complexity, that is if and only if

$$|\{(-x + G^{(\omega)}) \wedge B_r : x \in \mathcal{V}_G, G^{(\omega)} \in \widehat{\mathcal{G}}\}| < \infty \quad (3.3)$$

for every  $r > 0$ .

Replacing  $G$  and  $G'$  in the metric (2.3) by elements of  $\widehat{\mathcal{G}}$ , we obtain a metric on  $\widehat{\mathcal{G}}$ . This metric is used in the completion of the metric space

$$\widehat{X}_{\mathcal{G}} := \overline{\{x + G^{(\omega)} : x \in \mathbb{R}^d, G^{(\omega)} \in \widehat{\mathcal{G}}\}}. \quad (3.4)$$

An alternative description of the space  $\widehat{X}_{\mathcal{G}}$  is provided by

**Lemma 3.2.** *Let  $\mathcal{G}$  be a set of graphs with uniformly discrete vertex sets. Then*

$$\widehat{X}_{\mathcal{G}} = \{G^{(\omega)} : \omega \in \Omega_G, G \in X_{\mathcal{G}}\}. \quad (3.5)$$

*Proof.* To show the inclusion “ $\subseteq$ ”, it suffices to prove that the limit  $\widehat{G}$  of an arbitrary convergent sequence from  $\widehat{\mathcal{G}}$  is of the form  $G^{(\omega)}$  for some  $G \in X_{\mathcal{G}}$  and some  $\omega \in \Omega_G$ . So assume that for every  $\varepsilon > 0$  there exist  $x_\varepsilon \in \mathbb{R}^d$ ,  $y_\varepsilon \in B_\varepsilon$  and  $G_\varepsilon^{(\omega_\varepsilon)} \in \widehat{\mathcal{G}}$  such that  $(x_\varepsilon + G_\varepsilon^{(\omega_\varepsilon)}) \wedge B_{1/\varepsilon} = (y_\varepsilon + \widehat{G}) \wedge B_{1/\varepsilon}$ . Clearly, convergence of a sequence of coloured graphs implies convergence



of the underlying uncoloured graphs, that is, there exists  $G \in X_{\mathcal{G}}$  such that  $(x_\varepsilon + G_\varepsilon) \wedge B_{1/\varepsilon} = (y_\varepsilon + G) \wedge B_{1/\varepsilon}$  for all  $\varepsilon > 0$ . We define  $\omega \in \Omega_G$  as follows: given any  $\varepsilon > 0$ , we set  $\omega_e := \omega_{\varepsilon, y_\varepsilon - x_\varepsilon + e}$  for all  $e \in \mathcal{E}_{G \wedge B_{1/\varepsilon}}(-y_\varepsilon)$ . This choice is consistent in the sense that if  $0 < \varepsilon' < \varepsilon$ , then  $\omega_{\varepsilon, y_\varepsilon - x_\varepsilon + e} = \omega_{\varepsilon', y_{\varepsilon'} - x_{\varepsilon'} + e}$  for all  $e \in \mathcal{E}_{G \wedge B_{1/\varepsilon}}(-y_\varepsilon)$ . By choosing  $\varepsilon$  arbitrarily small, we obtain  $\omega_e$  for all  $e \in \mathcal{E}_G$ . It follows that  $\widehat{G} = G^{(\omega)}$ .

To show the inclusion “ $\supseteq$ ”, consider  $G^{(\omega)}$  for an arbitrary  $G \in X_{\mathcal{G}}$  and arbitrary  $\omega \in \Omega_G$ . Then, for every  $\varepsilon > 0$  there exist  $x_\varepsilon \in \mathbb{R}^d$ ,  $y_\varepsilon \in B_\varepsilon$  and  $G_\varepsilon \in \mathcal{G}$  such that  $(x_\varepsilon + G_\varepsilon) \wedge B_{1/\varepsilon} = (y_\varepsilon + G) \wedge B_{1/\varepsilon}$ . Define  $\omega_{\varepsilon, e} := \omega_{x_\varepsilon - y_\varepsilon + e}$  for all  $e \in \mathcal{E}_{G_\varepsilon \wedge B_{1/\varepsilon}}(-x_\varepsilon)$  and set  $\omega_{\varepsilon, e}$  to an arbitrary value for the remaining edges. We then have  $G_\varepsilon^{(\omega_\varepsilon)} \in \widehat{\mathcal{G}}$  for all  $\varepsilon > 0$  and  $x_\varepsilon + G_\varepsilon^{(\omega_\varepsilon)} \rightarrow G^{(\omega)}$  as  $\varepsilon \downarrow 0$ .  $\square$

There is an analogue to Lemma 2.3 in the previous section. Recalling Remark 3.1, it can be formulated as

**Lemma 3.3.** *Let  $\mathcal{G}$  be a set of graphs with uniformly discrete vertex sets. Then,  $\widehat{X}_{\mathcal{G}}$  is compact if and only if  $\mathcal{G}$  has finite local complexity.*  $\square$

The main goal of this section is to express an ergodic probability measure  $\widehat{\mu}$  on a compact space  $\widehat{X}_{\mathcal{G}}$  in terms of an ergodic probability measure  $\mu$  on  $X_{\mathcal{G}}$  and the probability measures  $\mathbb{P}_G$ . This will be achieved in Theorem 3.5 at the end of this section.

As a preparation for Theorem 3.5, we define cylinder sets of  $\widehat{X}_{\mathcal{G}}$  in analogy those of  $X_{\mathcal{G}}$  in the previous section. Given a pattern  $P^{(\eta)}$  of some coloured graph in  $\widehat{\mathcal{G}}$  and a Borel set  $U \subseteq \mathbb{R}^d$ , we set

$$\widehat{\Xi}_{P^{(\eta)}, U} := \{G^{(\omega)} \in \widehat{X}_{\mathcal{G}} : x + P^{(\eta)} \subseteq G^{(\omega)} \text{ for some } x \in U\}. \quad (3.6)$$

The basic step in the construction of an ergodic measure on  $\widehat{X}_{\mathcal{G}}$  is given by the following lemma, which extends [Ho3, Lemma 3.1] to graphs which are not necessarily aperiodic – and also to more general measures  $\mathbb{P}_G$ . We employ the notation  $\chi_S$  for the indicator function of some set  $S$ .

**Lemma 3.4.** *Let  $\mathcal{G}$  be a set of graphs of finite local complexity and with uniformly discrete vertex sets of radius  $r > 0$ . Let  $\mu$  be an ergodic probability measure on  $X_{\mathcal{G}}$ . Then, given any pattern  $P^{(\eta)}$  of some coloured graph in  $\widehat{\mathcal{G}}$  and any Borel set  $U \subseteq \mathbb{R}^d$  with diameter  $\text{diam}(U) < r$ , the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \int_{B_n} dx \chi_{\widehat{\Xi}_{P^{(\eta)}, U}}(x + G^{(\omega)}) = \mu(\Xi_{P, U}) \mathbb{P}_P(\eta) \quad (3.7)$$

*exists for  $\mu$ -a.a.  $G \in X_{\mathcal{G}}$  and for  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ . If  $\widehat{X}_{\mathcal{G}}$  is uniquely ergodic, then the limit (3.7) exists for all  $G \in X_{\mathcal{G}}$  and for  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ .*

*Proof.* It follows from the definition of cylinder sets that

$$\int_{B_n} dx \chi_{\widehat{\Xi}_{P^{(\eta)}, U}}(x + G^{(\omega)}) = \text{vol}(U) \nu(P^{(\eta)}|G^{(\omega)} \wedge B_n) + \mathcal{O}(n^{d-1}) \quad (3.8)$$

asymptotically as  $n \rightarrow \infty$ , see also the proof of [LeMS, Thm. 2.7] for the argument. We analyse the asymptotic behaviour of the expression

$$\frac{\nu(P^{(\eta)}|G^{(\omega)} \wedge B_n)}{\text{vol}(B_n)} \leq \frac{\nu(P|G \wedge B_n)}{\text{vol}(B_n)}. \quad (3.9)$$

Ergodicity of  $X_G$  implies

$$\lim_{n \rightarrow \infty} \frac{\nu(P|G \wedge B_n)}{\text{vol}(B_n)} = \nu(P) = \frac{\mu(\Xi_{P,U})}{\text{vol}(U)}, \quad (3.10)$$

for  $\mu$ -a.a.  $G \in X_G$ , resp. for all  $G \in X_G$  in the uniquely ergodic case, see Lemmas 2.5 (iii) and 2.5 (iv). This proves the statement, if  $\mu(\Xi_{P,U}) = 0$ . Otherwise,  $\nu(P|G \wedge B_n)$  grows unboundedly in  $n$ . In particular,  $\nu(P|G \wedge B_n) \neq 0$  for  $n \geq n_0$ . Thus, we can write

$$\frac{\nu(P^{(\eta)}|G^{(\omega)} \wedge B_n)}{\text{vol}(B_n)} = \frac{\nu(P|G \wedge B_n)}{\text{vol}(B_n)} \frac{\nu(P^{(\eta)}|G^{(\omega)} \wedge B_n)}{\nu(P|G \wedge B_n)} \quad (3.11)$$

for  $n \geq n_0$ . It suffices to show that the second factor on the r.h.s. of (3.11) converges to  $\mathbb{P}_P(\eta)$  as  $n \rightarrow \infty$  for  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ . The latter statement follows from the strong law of large numbers, which is applicable due to Kolmogorov's criterion [Bau]. This is obvious, if all translates  $P_j$ ,  $j \in \mathbb{N}$ , of  $P$  in  $G$  are pairwise overlap-free and so that colours are assigned in a stochastically independent way to different translates.

Otherwise, we partition the set  $\{P_j\}_{j \in \mathbb{N}}$  of all such translates into a finite number  $\Delta$  of (non-empty) subsets  $\{P_j\}_{j \in J_\alpha}$ ,  $\alpha = 1, \dots, \Delta$ , such that there are no mutual overlaps between the translates within each of these subsets. Here we have  $\emptyset \neq J_\alpha \subset \mathbb{N}$  for all  $\alpha = 1, \dots, \Delta$ ,  $\cup_{\alpha=1}^\Delta J_\alpha = \mathbb{N}$  and  $J_\alpha \cap J_{\alpha'} = \emptyset$  for all  $\alpha \neq \alpha'$ . The existence of such a partition may be seen by a graph-colouring argument: construct a graph  $\mathcal{T}$  such that each translate  $P_j$  defines one point in the vertex set of  $\mathcal{T}$ . Two vertices are adjacent in  $\mathcal{T}$ , if the corresponding translates overlap. Clearly, a vertex colouring of  $\mathcal{T}$  (with adjacent vertices having different colours) provides an example for the partition that we are seeking. Due to uniform discreteness, the degree of any vertex in  $\mathcal{T}$  is bounded by some number  $d_{\mathcal{T},\max} < \infty$ . Thus, the vertex-colouring theorem [D] ensures the existence of such a colouring with  $\Delta \leq 1 + d_{\mathcal{T},\max}$  different colours. Denoting the number of elements in  $\{P_j\}_{j \in J_\alpha}$  with the property  $P_j \wedge B_n = P_j$  by  $\nu_\alpha(P|G \wedge B_n)$ , and denoting by  $\nu_\alpha(P^{(\eta)}|G^{(\omega)} \wedge B_n)$  the analogous quantity requiring, in addition, a match of the edge colourings, we can write

$$\frac{\nu(P^{(\eta)}|G^{(\omega)} \wedge B_n)}{\nu(P|G \wedge B_n)} = \sum_{\alpha=1}^\Delta \frac{\nu_\alpha(P^{(\eta)}|G^{(\omega)} \wedge B_n)}{\nu_\alpha(P|G \wedge B_n)} \frac{\nu_\alpha(P|G \wedge B_n)}{\nu(P|G \wedge B_n)}. \quad (3.12)$$

Here we assume  $n$  large enough so that  $\nu_\alpha(P|G \wedge B_n) > 0$  for all  $\alpha$ . Clearly, those  $\alpha$  for which the index set  $J_\alpha$  is finite do not contribute to (3.12) in the macroscopic limit  $n \rightarrow \infty$ , because for them the right-most fraction in (3.12) vanishes in the limit. For the remaining  $\alpha$ 's, the strong law of large numbers can be applied and gives

$$\lim_{n \rightarrow \infty} \frac{\nu_\alpha(P^{(\eta)}|G^{(\omega)} \wedge B_n)}{\nu_\alpha(P|G \wedge B_n)} = \mathbb{P}_P(\eta) \quad (3.13)$$

for  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ . Therefore we conclude

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \left| \frac{\nu(P^{(\eta)} | G^{(\omega)} \wedge B_n)}{\nu(P | G \wedge B_n)} - \mathbb{P}_P(\eta) \right| \\
& \leq \lim_{n \rightarrow \infty} \sum_{\alpha=1}^{\Delta} \left| \frac{\nu_{\alpha}(P^{(\eta)} | G^{(\omega)} \wedge B_n)}{\nu_{\alpha}(P | G \wedge B_n)} - \mathbb{P}_P(\eta) \right| \frac{\nu_{\alpha}(P | G \wedge B_n)}{\nu(P | G \wedge B_n)} \\
& \leq \lim_{n \rightarrow \infty} \sum_{\substack{\alpha \in \{1, \dots, \Delta\}: \\ |J_{\alpha}| = \infty}} \left| \frac{\nu_{\alpha}(P^{(\eta)} | G^{(\omega)} \wedge B_n)}{\nu_{\alpha}(P | G \wedge B_n)} - \mathbb{P}_P(\eta) \right| \\
& = 0.
\end{aligned} \tag{3.14}$$

for  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ , and the lemma follows together with (3.8), (3.11) and (3.10).  $\square$

Having established Lemma 3.4, which is an extension of [Ho3, Lemma 3.1] to more general graphs, we can now argue as in the proof of [Ho3, Thm. 3.1] to obtain the central result of this section.

**Theorem 3.5.** *Let  $\mathcal{G}$  be a set of graphs of finite local complexity and with uniformly discrete vertex sets of radius  $r > 0$ . Fix an ergodic probability measure  $\mu$  on  $X_{\mathcal{G}}$ . Then there exists a unique ergodic probability measure  $\hat{\mu}$  on  $\hat{X}_{\mathcal{G}}$  such that*

(i) *for every pattern  $P^{(\eta)}$  of some coloured graph in  $\hat{\mathcal{G}}$  and every Borel set  $U \subseteq \mathbb{R}^d$  with diameter  $\text{diam}(U) < r$  the relation*

$$\hat{\mu}(\hat{\Xi}_{P^{(\eta)}, U}) = \mu(\Xi_{P, U}) \mathbb{P}_P(\eta) \tag{3.15}$$

*holds.*

(ii) *for every  $\phi \in L^1(\hat{X}_{\mathcal{G}}, \hat{\mu})$  the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \int_{B_n} dx \phi(x + G^{(\omega)}) = \int_{\hat{X}_{\mathcal{G}}} d\hat{\mu}(F^{(\sigma)}) \phi(F^{(\sigma)}) \tag{3.16}$$

*exist for  $\hat{\mu}$ -a.a.  $G^{(\omega)} \in \hat{X}_{\mathcal{G}}$ . If  $\hat{X}_{\mathcal{G}}$  is even uniquely ergodic and, in addition, if  $\phi$  is either continuous or a linear combination of cylinder functions, then the limit (3.16) exists for all  $G \in X_{\mathcal{G}}$  and for  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ .*

(iii) *for every  $\phi \in L^1(\hat{X}_{\mathcal{G}}, \hat{\mu})$  we have*

$$\int_{\hat{X}_{\mathcal{G}}} d\hat{\mu}(G^{(\omega)}) \phi(G^{(\omega)}) = \int_{X_{\mathcal{G}}} d\mu(G) \int_{\Omega_G} d\mathbb{P}_G(\omega) \phi(G^{(\omega)}). \tag{3.17}$$

$\square$

**Remarks 3.6.** (i) The corresponding theorem [Ho3, Thm. 3.1] is a statement about Bernoulli site percolation on the Penrose tiling. Our result is an extension, which covers both the aperiodic and the periodic situations, under weaker assumptions on the base graphs.

(ii) The asserted uniqueness of the ergodic measure  $\hat{\mu}$  in the theorem does not mean that the dynamical system  $\hat{X}_{\mathcal{G}}$  is uniquely ergodic. It only means that  $\hat{\mu}$  is unique for the given ergodic measure  $\mu$  on  $X_{\mathcal{G}}$  and the measures  $\mathbb{P}_G$  on  $\Omega_G$ .

## 4. FINITE-RANGE OPERATORS ON RANDOMLY COLOURED GRAPHS

In this section, we consider covariant finite-range operators on randomly coloured graphs, together with some of their basic ergodic and spectral properties. We ensure the existence of their integrated density of states, derive its self-averaging property, and study the non-randomness of the spectrum. For the spectral-theoretic background, the reader is referred to [RS1, RS2].

For a countable set  $\mathcal{V}$ , let  $\ell^2(\mathcal{V})$  be the Hilbert space of square-summable functions  $\psi : \mathcal{V} \rightarrow \mathbb{C}$  with canonical scalar product  $\langle \cdot, \cdot \rangle$ . We denote the canonical basis in  $\ell^2(\mathcal{V})$  by  $\{\delta_v\}_{v \in \mathcal{V}}$ , that is  $\delta_v(w) = 1$  if  $w = v$  and zero otherwise.

**Definition 4.1.** (i) Let  $G^{(\omega)}$  be a coloured graph. A bounded linear operator  $H_{G^{(\omega)}}$  in  $\ell^2(\mathcal{V}_{G^{(\omega)}})$  is said to be *covariant of range*  $R \in ]0, \infty[$ , if

- (1)  $\langle \delta_{x+v}, H_{G^{(\omega)}} \delta_{x+w} \rangle = \langle \delta_v, H_{G^{(\omega)}} \delta_w \rangle$  for all  $v, w \in \mathcal{V}_{G^{(\omega)}}$  and all  $x \in \mathbb{R}^d$  for which  $G^{(\omega)} \wedge (B_R(x+v) \cup B_R(x+w)) = G^{(\omega)} \wedge (B_R(v) \cup B_R(w))$ ,
- (2)  $\langle \delta_v, H_{G^{(\omega)}} \delta_w \rangle = 0$  for all  $v, w \in \mathcal{V}_{G^{(\omega)}}$  subject to  $|v - w| \geq R$ .

(ii) Let  $\mathcal{G}$  be a set of graphs of finite local complexity and with uniformly discrete vertex sets of radius  $r$ . Fix an ergodic probability measure  $\hat{\mu}$  on  $\hat{X}_{\mathcal{G}}$ . Given any  $R \in ]0, \infty[$ , we call a mapping  $\hat{H} : G^{(\omega)} \mapsto H_{G^{(\omega)}}$  from  $\hat{X}_{\mathcal{G}}$ , with values in the set of bounded, self-adjoint operators that are covariant of range  $R$ , a  *$\hat{\mu}$ -ergodic self-adjoint operator of finite range*.

**Remarks 4.2.** (i) The covariance condition in the above definition means that  $H_{G^{(\omega)}}$  is determined on the class of non-equivalent  $R$ -patterns.

(ii) A  $\hat{\mu}$ -ergodic self-adjoint operator of finite range  $\hat{H}$  is *uniformly bounded*, in the sense that  $\sup_{G^{(\omega)} \in \hat{X}_{\mathcal{G}}} \|H_{G^{(\omega)}}\| < \infty$ , where  $\|\cdot\|$  denotes the usual operator norm. In particular, there exists a compact interval  $K \subset \mathbb{R}$ , such that for  $\hat{\mu}$ -almost every  $G^{(\omega)} \in \hat{X}_{\mathcal{G}}$  the spectrum of  $H_{G^{(\omega)}}$  is contained in  $K$ .

The eigenvalue density of  $\hat{H}$  is a quantity of great interest in applications.

**Definition 4.3.** Let  $\hat{H}$  be a  $\hat{\mu}$ -ergodic self-adjoint operator of finite range, and fix a mollifier as in Corollary 2.5 (i). Then, the *integrated density of states* of  $\hat{H}$  is defined as the right-continuous distribution function  $\mathbb{R} \rightarrow [0, \varrho]$ ,

$$E \mapsto N(E) := \int_{\hat{X}_{\mathcal{G}}} d\hat{\mu}(G^{(\omega)}) \sum_{v \in \mathcal{V}_{G^{(\omega)}}} \psi(v) \langle \delta_v, \Theta(E - H_{G^{(\omega)}}) \delta_v \rangle. \quad (4.1)$$

In (4.1) we have denoted the right-continuous Heaviside unit-step function by  $\Theta := \chi_{[0, \infty[}$  and the mollifier  $\psi$  was introduced in Corollary 2.5 (i).

**Remarks 4.4.** (i) The integrand in (4.1) is bounded, see Remark 2.6 (i). Moreover, it is measurable. In fact, the map  $\hat{X}_{\mathcal{G}} \rightarrow \mathbb{C}$ ,

$$G^{(\omega)} \mapsto f_{\psi}(G^{(\omega)}) := \sum_{v \in \mathcal{V}_{G^{(\omega)}}} \psi(v) \langle \delta_v, f(H_{G^{(\omega)}}) \delta_v \rangle \quad (4.2)$$

is even continuous for all  $f \in L^\infty(\mathbb{R})$ , thanks to the continuity of  $\psi$ .

(ii) The unique Borel measure  $dN$  on  $\mathbb{R}$  associated with the distribution function  $N$  has a total mass, given by the vertex density  $\varrho$ , cf. Corollary 2.5 (i). Moreover,  $dN$  is compactly supported, due to the boundedness of  $\hat{H}$ .

(iii) Clearly, the integrated density of states  $N$  depends on the choice of the ergodic measure  $\hat{\mu}$  on  $\hat{X}_G$ . However, ergodicity of  $\hat{\mu}$  implies that  $N$  does not depend on the choice of the mollifier  $\psi$ . This will become manifest in Theorem 4.5 below.

The integrated density of states of  $\hat{H}$  can also be characterised in terms of a macroscopic limit.

**Theorem 4.5.** *Let  $\hat{H}$  be a  $\hat{\mu}$ -ergodic, self-adjoint operator of finite range and let  $N$  be its integrated density of states (4.1), for some choice of the mollifier  $\psi$ . Then there exists a Borel set  $\hat{A} \subseteq \hat{X}_G$  of full probability,  $\hat{\mu}(\hat{A}) = 1$ , such that*

$$N(E) = \lim_{n \rightarrow \infty} \left\{ \frac{1}{\text{vol}(B_n)} \sum_{v \in \mathcal{V}_{G(\omega)} \cap B_n} \langle \delta_v, \Theta(E - H_{G(\omega)}) \delta_v \rangle \right\} \quad (4.3)$$

*holds for all  $G^{(\omega)} \in \hat{A}$  and all  $E \in \mathbb{R}$ , except for the (at most countably many) discontinuity points of  $N$ . If  $\hat{X}_G$  is uniquely ergodic, then convergence holds even for all  $G \in X_G$  and  $\mathbb{P}_G$ -a.a.  $\omega \in \Omega_G$ .*

*Sketch of the proof.* The theorem follows from vague convergence of the associated measures by standard arguments [Bau, Thms. 30.8, 30.13]. Vague convergence follows in turn from the Ergodic Theorem 3.5 (ii), and from the identity

$$\sum_{v \in \mathcal{V}_{G(\omega)} \cap B_n} \langle \delta_v, f(H_{G(\omega)}) \delta_v \rangle = \int_{B_n} dx f_\psi(x + G^{(\omega)}) + \mathcal{O}(n^{d-1}), \quad (4.4)$$

which is valid for arbitrary  $f \in C_c(\mathbb{R})$  and arbitrary mollifiers  $\psi$ . The continuous function  $f_\psi$ , associated with  $f$ , was defined in Remark 4.4 (i).  $\square$

**Remark 4.6.** For systems of randomly coloured subgraphs of  $\mathbb{Z}^d$  one can even prove uniform convergence in the energy  $E$  [LMV]. Such a result, which is based on ideas in [LS2], has now been generalised in [LV]. In particular, it applies also to random colourings of graphs with Delone vertex sets. In addition, the origin of the discontinuities of  $N$  is related to compactly supported eigenfunctions and their sizes to equivariant dimensions [LV].

Statements corresponding to Theorem 4.5 for systems of uncoloured Delone sets with finite local complexity can be found in [Ho2], [LS1, Prop. 4.6], [LS2, Thm. 3] and [LPV, Thm. 6.1]. The papers [Ho2] and [LS2] deal with strictly ergodic systems, for which [LS2] establish even uniform convergence in the energy  $E$ .

Next, we relate the set of growth points of the integrated density of states  $N$  to the spectrum of  $\hat{H}$ . Given any a self-adjoint operator  $A$ , we denote its

spectrum by  $\text{spec}(A)$  and the essential part of the spectrum by  $\text{spec}_{\text{ess}}(A)$ . We write  $\text{supp}(\text{d}N)$  for the topological support of the probability measure on  $\mathbb{R}$ , whose distribution function is the integrated density of states  $N$ . The topological support can be characterised as

$$\text{supp}(\text{d}N) = \left\{ E \in \mathbb{R} : \int_{]\lambda, \lambda'[} \text{d}N > 0 \quad \text{for all } \lambda, \lambda' \in \mathbb{Q} \text{ with } \lambda < E < \lambda' \right\}. \quad (4.5)$$

**Theorem 4.7.** *Let  $\widehat{H}$  be a  $\widehat{\mu}$ -ergodic, self-adjoint operator of finite range. Then there exists a Borel set  $\overline{X} \subseteq X_G$  with  $\mu(\overline{X}) = 1$  such that*

$$\text{spec}(H_{G(\omega)}) = \text{spec}_{\text{ess}}(H_{G(\omega)}) = \text{supp}(\text{d}N) \quad (4.6)$$

for all  $G \in \overline{X}$  and  $\mathbb{P}_G$ -almost all  $\omega \in \Omega_G$ . Moreover, if  $X_G$  is uniquely ergodic and obeys the positive lower frequency condition, see Definition 2.7, then the statement holds even with  $\overline{X} = X_G$ , that is, for every  $G \in X_G$ .

The theorem follows from Lemmas 6.4 and 6.5 below.

**Remark 4.8.** The most interesting part of Theorem 4.7 is that the statement (4.6) holds for all  $G \in X_G$ , under the stronger assumptions of unique ergodicity and the positive lower frequency condition. For uncoloured Delone systems, such a result can be found in [Ho1, Prop. 7.4] and [LS1, Lemma 3.6, Thm. 4.3], cf. Remarks 2.8.

The weaker  $\mu$ -almost sure statement of Theorem 4.7, which holds without the additional hypotheses, is analogous to results in [LPV], where they are proved in the context of ergodic groupoids.

## 5. LIFSHTS TAILS FOR THE LAPLACIAN ON BOND-PERCOLATION GRAPHS

In this section, we study the asymptotics at the lower spectral edge of the integrated density of states of graph Laplacians, which are associated to (Bernoulli) bond-percolation subgraphs of a given graph. We will specialise the general framework of the previous sections in three respects.

First, we set  $\mathcal{G} := \{G_0\}$ , where  $G_0$  is some fixed graph of finite local complexity, with a uniformly discrete vertex set and a bounded degree sequence. For notational simplicity we write  $X$  instead of  $X_{\{G_0\}}$ , and  $\widehat{X}$  instead of  $\widehat{X}_{\{G_0\}}$ . We fix an ergodic measure  $\mu$  on  $X$ .

Second, we interpret a Bernoulli bond-percolation subgraph as a particular randomly coloured graph. To this end, we choose the set of colours  $\mathbb{A} = \{0, 1\}$  and make the identification of a coloured graph  $(G, \omega) \in \widehat{X}$  with the subgraph  $(\mathcal{V}_G, \mathcal{E}_G^{(\omega)})$  of  $G$  which has the same vertex set as  $G$  and edge set  $\mathcal{E}_G^{(\omega)} := \{e \in \mathcal{E}_G : \omega_e = 1\}$ . We denote this subgraph again by  $G^{(\omega)}$ . The probability measure on  $\Omega_G$  is given by the  $|\mathcal{E}_G|$ -fold product measure  $\mathbb{P}_G^p = \bigotimes_{e \in \mathcal{E}_G} \mathbb{P}_0^p$  of the Bernoulli measure  $\mathbb{P}_0^p := p\delta_1 + (1-p)\delta_0$  on  $\mathbb{A}$  with parameter  $p \in [0, 1]$ . Thus, each edge is taken away from  $G$  independently of the others with probability  $1-p$ . According to Section 3, the ergodic measure  $\mu$  on  $X$  gives rise to an ergodic measure  $\widehat{\mu}_p$  on  $\widehat{X}$ .

Third, we take the operator  $H_{G^{(\omega)}}$  to be the combinatorial or graph Laplacian  $\Delta_{G^{(\omega)}}$  associated with the graph  $G^{(\omega)}$ . The graph Laplacian is a covariant operator of range  $R = 2$  in the sense of the previous section, as follows easily from its definition in

**Lemma 5.1.** *Given an arbitrary graph  $G = (\mathcal{V}, \mathcal{E})$  with bounded degree sequence, the graph Laplacian  $\Delta_G : \ell^2(\mathcal{V}) \rightarrow \ell^2(\mathcal{V})$ ,*

$$(\Delta_G \varphi)(v) := \sum_{w \in \mathcal{V} : \{v, w\} \in \mathcal{E}} [\varphi(v) - \varphi(w)] \quad (5.1)$$

*for all  $\varphi \in \ell^2(\mathcal{V})$  and all  $v \in \mathcal{V}$ , is a bounded, self-adjoint and non-negative linear operator. Moreover, zero is an eigenvalue of  $\Delta_G$ , if and only if  $G$  possesses a finite cluster. The multiplicity of the eigenvalue zero is given by the number of finite clusters of  $G$ .  $\square$*

We omit the straightforward proof of the lemma and refer to standard accounts [Ch, Co] on spectral graph theory instead. The mapping  $\hat{X} \ni G^{(\omega)} \mapsto \Delta_{G^{(\omega)}}$  defines a  $\hat{\mu}_p$ -ergodic, self-adjoint operator of finite range in the sense of Definition 4.1 (ii), the *Laplacian on bond-percolation graphs associated with  $G_0$* .

The graph  $G_0$  need not be connected in what follows. However, it is crucial that the vertex density  $\varrho_\infty$  of its infinite component(s), which was introduced in Corollary 2.5 (ii), is strictly positive.

**Theorem 5.2.** *Let  $G_0$  be a graph of finite local complexity, with a uniformly discrete vertex set, a bounded degree sequence  $d_{\max} := \sup_{v \in \mathcal{V}} d_{G_0}(v) < \infty$ , and a maximal edge length  $l_{\max} := \sup\{|u - v| : \{u, v\} \in \mathcal{E}\} < \infty$ . Assume further that  $\varrho_\infty > 0$ , with respect to some ergodic measure  $\mu$  on  $X$ . Consider the Laplacian on bond-percolation graphs associated with  $G_0$ , and let  $N$  be its integrated density of states (4.1), with respect to the measure  $\hat{\mu}_p$ . If  $p \in ]0, \frac{1}{d_{\max}-1}[$ , then*

$$\lim_{E \downarrow 0} \frac{\ln |\ln[N(E) - N(0)]|}{\ln E} = -1/2, \quad (5.2)$$

*that is,  $N$  exhibits a Lifshits tail with Lifshits exponent  $1/2$  at the lower edge of the spectrum.*

**Remark 5.3.** (i) Theorem 5.2 follows from Lemmas 5.4 and 5.5 below, which provide slightly stronger statements as needed to conclude (5.2). The lemmas show that Theorem 5.2 holds for all edge probabilities  $p$ , for which the cluster-size distribution decays exponentially for  $\mu$ -almost every graph  $G \in X$ . This means in particular, that the validity of Theorem 5.2 is limited to the *non-percolating phase*, that is  $\{p \in [0, 1] : \sup_{v \in \mathcal{V}_{G_0}} \mathbb{P}_{G_0}^p(|C_v| = \infty) = 0\}$ , see Section 7. Exponential decay of the cluster-size distribution is guaranteed by the conditions  $l_{\max} < \infty$  and  $p \in ]0, \frac{1}{d_{\max}-1}[$ , see Corollary 7.5.

(ii) Remark 2.6 (iii) states sufficient conditions for  $\varrho_\infty > 0$ .

(iii) Theorem 5.2 generalises part of Theorem 1.14 in [KiM], where the special case  $G_0 = \mathbb{L}^d$  of the  $d$ -dimensional integer lattice was considered.

(iv) The Lifshits exponent  $1/2$  in (5.2) does not depend on the spatial dimension  $d$  of the underlying space. This comes from the fact that the asymptotics (5.2) is determined by the longest linear clusters of the percolation graphs  $G^{(\omega)} \in \widehat{X}$ .

**Lemma 5.4.** *Let  $G_0$  be a graph of finite local complexity and with a uniformly discrete vertex set. Assume there exists  $p_0 \in ]0, 1[$  such that for every  $p \in [0, p_0[$  and  $\mu$ -almost all  $G \in X$  the cluster-size distribution for percolation on  $G$  decays exponentially, i.e.,*

$$\mathbb{P}_G^p\{\omega \in \Omega_G : |C_v^{(\omega)}| \geq n\} \leq D(p) \exp\{-\lambda(p)n\} \quad (5.3)$$

for all  $n \in \mathbb{N}$ , where  $D(p), \lambda(p) \in ]0, \infty[$  are constants that depend on  $p$ , but are uniform in  $G \in X$  and  $v \in \mathcal{V}_G$ . Here  $C_v^{(\omega)}$  denotes the cluster of the graph  $G^{(\omega)}$  containing  $v \in \mathcal{V}_G$ . Then

$$N(E) - N(0) \leq \varrho D(p) \exp\{-\lambda(p)E^{-1/2}\} \quad (5.4)$$

for all  $p \in [0, p_0[$  and all  $E > 0$ . The vertex density  $\varrho$  was introduced in (2.9).

*Proof.* The proof is analogous to that of Lemma 2.7 (Neumann case) in [KiM]. The block-diagonal form of the Laplacian with respect to the cluster structure implies

$$\begin{aligned} N(E) - N(0) &= \int_X d\mu(G) \sum_{v \in \mathcal{V}_G} \psi(v) \int_{\Omega_G} d\mathbb{P}_G^p(\omega) \\ &\quad \times \langle \delta_v, [\Theta(E - \Delta_{C_v^{(\omega)}}) - \Theta(\Delta_{C_v^{(\omega)}})] \delta_v \rangle \\ &\leq \int_X d\mu(G) \sum_{v \in \mathcal{V}_G} \psi(v) \int_{\Omega_G} d\mathbb{P}_G^p(\omega) \Theta(E - E_1(C_v^{(\omega)})) \\ &\quad \times \langle \delta_v, [\mathbf{1} - \Theta(\Delta_{C_v^{(\omega)}})] \delta_v \rangle \\ &\leq \int_X d\mu(G) \sum_{v \in \mathcal{V}_G} \psi(v) \mathbb{P}_G^p\{\omega \in \Omega_G : E \geq E_1(C_v^{(\omega)})\}, \end{aligned} \quad (5.5)$$

where  $E_1(C_v^{(\omega)})$  denotes the smallest non-zero eigenvalue of the Laplacian on the cluster  $C_v^{(\omega)}$ . As a particular consequence of the decay (5.3) of the cluster-size distribution, we infer that  $|C_v^{(\omega)}| < \infty$  for all  $v \in \mathcal{V}_G$  holds for  $\mu$ -almost all  $G \in X$  and  $\mathbb{P}_G^p$ -almost all  $\omega \in \Omega$ . Hence, Cheeger's inequality  $E_1(C_v^{(\omega)}) \geq |C_v^{(\omega)}|^{-2}$ , see e.g. Lemma A.1 in [KhKM], can be applied to estimate the probability in (5.5), and we get

$$N(E) - N(0) \leq \int_X d\mu(G) \sum_{v \in \mathcal{V}_G} \psi(v) \mathbb{P}_G^p\{\omega \in \Omega_G : |C_v^{(\omega)}| \geq E^{-1/2}\}. \quad (5.6)$$

The lemma now follows from the exponential decay (5.3) of the cluster-size distribution and from (2.9).  $\square$

**Lemma 5.5.** *Let  $G_0$  be a graph of finite local complexity, with a uniformly discrete vertex and bounded degree sequence  $d_{\max} := \sup_{v \in \mathcal{V}} d_{G_0}(v) < \infty$ .*



Let  $p \in ]0, 1[$  and  $E > 0$ . Then

$$N(E) - N(0) \geq \varrho_\infty e^{-2\gamma(p)} \frac{\exp\{-4\gamma(p) E^{-1/2}\}}{2 + 4E^{-1/2}}, \quad (5.7)$$

where  $\gamma(p) := -\ln p - d_{\max} \ln(1 - p) > 0$ .

*Proof.* We adapt the strategy of the proof of Lemma 2.9 (Neumann case) in [KiM]. In the present setting, we have to cope with the additional difficulty that vertices in  $G$  which are connected can be very far apart in the Euclidean metric.

Fix  $E > 0$  arbitrary and let  $\{\varepsilon_j\}_{j \in \mathbb{N}}$  be a null sequence of positive reals, such that  $E + \varepsilon_j$  is a point of continuity of the integrated density of states  $N$  for all  $j \in \mathbb{N}$ . Then the right-continuity of  $N$ , the Ergodic Theorem 4.5 and the isotony of  $N$  imply

$$\begin{aligned} N(E) - N(0) &= \lim_{j \rightarrow \infty} [N(E + \varepsilon_j) - N(0)] \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \sum_{v \in \mathcal{V}_G} \chi_{B_n}(v) \langle \delta_v, [\Theta(E - \Delta_{G^{(\omega)}}) - \Theta(\Delta_{G^{(\omega)}})] \delta_v \rangle \end{aligned} \quad (5.8)$$

for  $\hat{\mu}_p$ -almost all graphs  $G^{(\omega)} \in \hat{X}$ . Since  $\Delta_{G^{(\omega)}}$  is a direct sum of the Laplacians of the clusters of  $G^{(\omega)}$ , and since this is also true for functions of the Laplacian, it follows that the trace on the right-hand side of (5.8) can be bounded from below by throwing away all contributions from branched clusters in that sum,

$$\begin{aligned} \sum_{v \in \mathcal{V}_G} \chi_{B_n}(v) \langle \delta_v, [\Theta(E - \Delta_{G^{(\omega)}}) - \Theta(\Delta_{G^{(\omega)}})] \delta_v \rangle \\ \geq \sum_{l=2}^{\infty} Z_{B_n}^{G^{(\omega)}}(\mathcal{L}_l) \langle \delta_1, [\Theta(E - \Delta_{\mathcal{L}_l}) - \Theta(\Delta_{\mathcal{L}_l})] \delta_1 \rangle. \end{aligned} \quad (5.9)$$

Here  $\mathcal{L}_l$  denotes a linear chain (i.e. non-branched and cycle-free cluster) with  $l$  vertices,  $Z_{B_n}^{G^{(\omega)}}(\mathcal{L}_l)$  the number of such chains in the percolation graph  $G^{(\omega)}$ , subject to the condition that at least one of its end-vertices lies in the ball  $B_n$ . The symbol  $\delta_1$  denotes the canonical basis vector in  $\ell^2(\{1, \dots, l\})$  corresponding to one end-vertex of  $\mathcal{L}_l$  (by symmetry reasons it does not matter which end-vertex).

The spectral representation of  $\Delta_{\mathcal{L}_l}$  with  $l \geq 2$  is explicitly known, for example, by mapping the problem to that of a cycle graph with  $2l$  vertices. The eigenvalues turn out to be

$$E_k(\mathcal{L}_l) := 4(\sin(\pi k/2l))^2, \quad k = 0, \dots, l, \quad (5.10)$$

and the components of the corresponding normalised eigenvectors  $\varphi_k$  in the canonical basis are given by  $\langle \delta_j, \varphi_0 \rangle := l^{-1/2}$  and

$$\langle \delta_j, \varphi_k \rangle := (2/l)^{1/2} \cos\left(\pi \frac{k}{l} \left(j - \frac{1}{2}\right)\right), \quad k = 1, \dots, l, \quad (5.11)$$

where  $j = 1, \dots, l$ . We observe that

$$\Theta(E - \Delta_{\mathcal{L}_l}) - \Theta(\Delta_{\mathcal{L}_l}) \geq \Theta(E - E_1(\mathcal{L}_l)) \varphi_1 \otimes \varphi_1, \quad (5.12)$$

where the dyadic product is the projector on the eigenspace generated by  $\varphi_1$ , and that  $E_1(\mathcal{L}_l) \leq 10/l^2$  and  $|\langle \delta_1, \varphi_1 \rangle|^2 \geq l^{-1}$ . Therefore we obtain

$$\begin{aligned} N(E) - N(0) &\geq \limsup_{n \rightarrow \infty} \sum_{l=2}^{\infty} \Theta(E - 10/l^2) \frac{1}{l} \frac{Z_{B_n}^{G^{(\omega)}}(\mathcal{L}_l)}{\text{vol}(B_n)} \\ &\geq \frac{1}{l(E)} \limsup_{n \rightarrow \infty} \frac{Z_{B_n}^{G^{(\omega)}}(\mathcal{L}_{l(E)})}{\text{vol}(B_n)} \end{aligned} \quad (5.13)$$

for  $\hat{\mu}_p$ -almost all graphs  $G^{(\omega)} \in \hat{X}$  with  $l(E) := \inf\{l \in \mathbb{N} \setminus \{1\} : E - 10/l^2 \geq 0\}$ .

The quantity

$$g_v(G^{(\omega)}) := \begin{cases} 1, & \text{if a vertex } v \in \mathcal{V}_G \text{ is an end-vertex of a linear chain} \\ & \text{with } l(E) \text{ vertices in } G^{(\omega)} \\ 0, & \text{otherwise} \end{cases} \quad (5.14)$$

helps to rewrite the right-hand side of (5.13), so that

$$\begin{aligned} N(E) - N(0) &\geq \frac{1}{2l(E)} \limsup_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \sum_{v \in \mathcal{V}_G} \chi_{B_n(v)} g_v(G^{(\omega)}) \\ &= \frac{1}{2l(E)} \limsup_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \int_{B_n} dx \sum_{v \in \mathcal{V}_{x+G}} \psi(v) g_v(x + G^{(\omega)}) \end{aligned} \quad (5.15)$$

for  $\hat{\mu}_p$ -almost all graphs  $G^{(\omega)} \in \hat{X}$ . The Ergodic Theorem 3.5 (ii), (iii) now implies

$$N(E) - N(0) \geq \frac{1}{2l(E)} \int_X d\mu(G) \sum_{v \in \mathcal{V}_G} \psi(v) \int_{\Omega_G} d\mathbb{P}_G^p(\omega) g_v(G^{(\omega)}). \quad (5.16)$$

We recall that  $\chi_{\mathcal{V}_{G,\infty}}(v) = 1$ , if  $v$  belongs to an infinite component of  $G$ , and zero otherwise. Then we have for all  $G \in X$  the crude elementary combinatorial estimate

$$\int_{\Omega_G} d\mathbb{P}_G^p(\omega) g_v(G^{(\omega)}) \geq 2p^{l(E)}(1-p)^{l(E)d_{\max}} \chi_{\mathcal{V}_{G,\infty}}(v) \quad (5.17)$$

for the probability that a given vertex appears as the end vertex of a linear chain with  $l(E)$  vertices. Here we have also used Lemma 2.2 (i). The estimate (5.17) yields

$$N(E) - N(0) \geq \frac{e^{-l(E)\gamma(p)}}{l(E)} \int_X d\mu(G) \sum_{v \in \mathcal{V}_G} \psi(v) \chi_{\mathcal{V}_{G,\infty}}(v) = \varrho_{\infty} \frac{e^{-l(E)\gamma(p)}}{l(E)}. \quad (5.18)$$

Making use of  $l(E) < 2 + 4E^{-1/2}$ , we obtain the assertion of the lemma.  $\square$

## 6. PROOF OF THEOREM 4.7

Lemmas 6.4 and 6.5 in this section provide the proof of Theorem 4.7. We begin with a standard result on the non-randomness of the spectrum.

**Lemma 6.1.** *Let  $\widehat{H}$  be a  $\widehat{\mu}$ -ergodic, self-adjoint operator. Then there exists a Borel set  $\Sigma \subseteq \mathbb{R}$ , such that  $\text{spec}(H_{G^{(\omega)}}) = \Sigma$  for  $\widehat{\mu}$ -almost all  $G^{(\omega)} \in \widehat{X}_G$ .  $\square$*

**Remarks 6.2.** (i) The proof of the lemma is classical in the theory of random Schrödinger operators, see e.g. [CL, PF]. The central point in the argument is that for every Borel set  $I \subseteq \mathbb{R}$  the measurable function

$$\widehat{X}_G \ni G^{(\omega)} \mapsto t_I(G^{(\omega)}) := \text{tr } \chi_I(H_{G^{(\omega)}}) \quad (6.1)$$

is invariant under translations of  $G^{(\omega)}$  by arbitrary  $a \in \mathbb{R}^d$ , and hence  $\widehat{\mu}$ -almost surely constant by ergodicity.

(ii) Standard arguments extend the non-randomness also to the Lebesgue components of the spectrum [CL, PF]. These arguments are taken up in [LPV] in the context of ergodic groupoids.

**Lemma 6.3.** *Let  $\widehat{H}$  be a  $\widehat{\mu}$ -ergodic, self-adjoint operator of finite range, and let  $N$  be its integrated density of states. For a given open interval  $I := ]\lambda, \lambda'[,$  where  $\lambda, \lambda' \in \mathbb{R}$  with  $\lambda < \lambda'$ , consider the following statements.*

- (i)  $\int_I dN > 0$ .
- (ii) *there exists a Borel set  $\overline{X} \subseteq X_G$  with  $\mu(\overline{X}) = 1$  ( $\overline{X} = X_G$ , if  $X_G$  is uniquely ergodic) such that  $t_I(G^{(\omega)}) = \infty$  for all  $G \in \overline{X}$  and for  $\mathbb{P}_G$ -almost all  $\omega \in \Omega_G$ .*
- (iii) *there exists a Borel set  $\overline{X} \subseteq X_G$  with  $\mu(\overline{X}) = 1$  ( $\overline{X} = X_G$ , if  $X_G$  is uniquely ergodic) such that  $t_I(G^{(\omega)}) > 0$  for all  $G \in \overline{X}$  and for  $\mathbb{P}_G$ -almost all  $\omega \in \Omega_G$ .*
- (iv) *there exists a Borel set  $\underline{X} \subseteq X_G$  with  $\mu(\underline{X}) > 0$  such that  $t_I(G^{(\omega)}) > 0$  for all  $G \in \underline{X}$  and all  $\omega$  in some subset of  $\Omega_G$  that has a positive  $\mathbb{P}_G$ -measure.*
- (v) *there exists  $G \in X_G$  and  $\omega \in \Omega_G$  such that  $t_I(G^{(\omega)}) > 0$ .*

Then the implications

$$(i) \iff (ii) \iff (iii) \iff (iv) \implies (v) \quad (6.2)$$

hold. Moreover, if  $X_G$  is uniquely ergodic and obeys the positive lower frequency condition, then

$$(v) \implies (i) \quad (6.3)$$

holds, too.

*Proof.* (i)  $\implies$  (ii): Using the Ergodic Theorem 4.5, we deduce the existence of a Borel set  $\overline{X} \subseteq X_G$  with  $\mu(\overline{X}) = 1$  (and  $\overline{X} = X_G$ , if  $X_G$  is uniquely ergodic) such that

$$\begin{aligned} 0 &< \int_I dN = \int_{X_G} d\mu(\tilde{G}) \int_{\Omega_{\tilde{G}}} d\mathbb{P}_{\tilde{G}}(\tilde{\omega}) \sum_{v \in \mathcal{V}_{\tilde{G}}} \psi(v) \langle \delta_v, \chi_I(H_{\tilde{G}^{(\tilde{\omega})}}) \delta_v \rangle \\ &= \lim_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \sum_{v \in \mathcal{V}_G \cap B_n} \langle \delta_v, \chi_I(H_{G^{(\omega)}}) \delta_v \rangle \end{aligned} \quad (6.4)$$

for all  $G \in \overline{X}$  and for  $\mathbb{P}_G$ -almost all  $\omega \in \Omega_G$ . Hence,

$$t_I(G^{(\omega)}) = \lim_{n \rightarrow \infty} \sum_{v \in \mathcal{V}_G \cap B_n} \langle \delta_v, \chi_I(H_{G^{(\omega)}}) \delta_v \rangle = \infty \quad (6.5)$$

for those  $G^{(\omega)}$ .

(ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v): These implications are obvious.

(iv)  $\Rightarrow$  (i): Assume (i) is wrong. Then we have

$$0 = \int_I dN = \int_{X_G} d\mu(G) \int_{\Omega_G} d\mathbb{P}_G(\omega) \sum_{v \in \mathcal{V}_G} \psi(v) \langle \delta_v, \chi_I(H_{G^{(\omega)}}) \delta_v \rangle. \quad (6.6)$$

Note that, by translation invariance of the measure  $\mu$ , we can replace the mollifier  $\psi$  in (6.6) by  $\psi_a := \psi(\cdot - a)$ , for any translation vector  $a \in \mathbb{R}^d$ . Consequently, there exists a Borel set  $\overline{X}_a \subseteq X_G$  with  $\mu(\overline{X}_a) = 1$ , such that for all  $G \in \overline{X}_a$  there is  $\overline{\Omega}_{G,a} \subseteq \Omega_G$  measurable with  $\mathbb{P}_G(\overline{\Omega}_{G,a}) = 1$ , such that for all  $\omega \in \overline{\Omega}_{G,a}$  we have  $\langle \delta_v, \chi_I(H_{G^{(\omega)}}) \delta_v \rangle = 0$ , for all  $v \in \mathcal{V}_G \cap \text{supp}(\psi_a)$ . We can choose a countable set  $M$  of translation vectors, such that for every  $v \in \mathcal{V}_G$  there exists  $a \in M$  with  $v \in \text{supp} \psi_a$ . Next, we define  $\overline{X} := \bigcap_{a \in M} \overline{X}_a$  and  $\overline{\Omega}_G := \bigcap_{a \in M} \overline{\Omega}_{G,a}$  for all  $G \in \overline{X}$  so that  $\mu(\overline{X}) = 1$  and  $\mathbb{P}_G(\overline{\Omega}_G) = 1$  for all  $G \in \overline{X}$ . Now, we get  $\langle \delta_v, \chi_I(H_{G^{(\omega)}}) \delta_v \rangle = 0$  for all  $G \in \overline{X}$ , all  $\omega \in \overline{\Omega}_G$  and all  $v \in \mathcal{V}_G$ . In other words,  $\chi_I(H_{G^{(\omega)}}) = 0$  for all  $G \in \overline{X}$  and all  $\omega \in \overline{\Omega}_G$ . This contradicts (iv) so that the implication is proven.

(v)  $\Rightarrow$  (i): By assumption,  $I$  contains a spectral value of  $H_{G^{(\omega)}}$ . Therefore there exists  $E \in I$ ,  $\delta \in ]0, \varepsilon[$ , where  $\varepsilon := \text{dist}(E, \mathbb{R} \setminus I)$ , and  $\varphi \in \ell^2(\mathcal{V}_G)$ ,  $\varphi \neq 0$ , such that

$$\|(H_{G^{(\omega)}} - E)\varphi\| < \delta \|\varphi\|. \quad (6.7)$$

Since  $H_{G^{(\omega)}}$  is bounded, we can even assume that  $\varphi$  has compact support. Furthermore, since  $\widehat{H}$  is of finite range  $R$ , we choose a compact subset  $K_0$  of  $\mathbb{R}^d$  such that  $\text{dist}(\text{supp}(\varphi), \mathbb{R}^d \setminus K_0) > 2R$ . We write  $P_0 := G \wedge K_0$  for the corresponding pattern of  $G$ . From the positive lower frequency condition we infer that the copies  $P_j := a_j + P_0$ ,  $j \in \mathbb{N}$ ,  $a_j \in \mathbb{R}^d$ , of this pattern in  $G$  occur with a positive lower frequency. For any given  $P_j$  there is a maximum number (which is uniform in  $j$ ) of other copies  $P_{j'}$  with which  $P_j$  can overlap. Therefore, from now on, we will pass to a subsequence of the sequence  $\{P_j\}_{j \in \mathbb{N}_0}$  such that none of the patches in the subsequence overlap, and still

$$\liminf_{n \rightarrow \infty} \frac{\tilde{\nu}(P_0 | G \wedge B_n)}{\text{vol}(B_n)} =: \gamma > 0. \quad (6.8)$$

Here the symbol  $\tilde{\nu}$  is used instead of  $\nu$  to indicate that it is only the patterns in the subsequence which are counted in  $G \wedge B_n$ . We denote the subsequence again by  $\{P_j\}_{j \in \mathbb{N}_0}$  and introduce the translated functions  $\varphi_j := \varphi(\cdot - a_j)$  for  $j \in \mathbb{N}$ . They form an orthogonal sequence, because  $\text{supp}(\varphi_j) \subset K_j := a_j + K_0$ , and the  $K_j$ 's are pairwise disjoint. Next we have to ensure that the colouring of  $G^{(\omega)}$  in  $K_0$  is also repeated in sufficiently many of the  $K_j$ . The events

$$A_j := \{\overline{\omega} \in \Omega_G : \overline{\omega}_{a_j+v} = \omega_v \text{ for all } v \in \mathcal{V}_G \cap K_0\}, \quad (6.9)$$

$j \in \mathbb{N}_0$ , where  $a_0 := 0$ , are all independent and  $\mathbb{P}_G(A_j) = \mathbb{P}_G(A_0) > 0$  for all  $j \in \mathbb{N}$ . Thus, the strong law of large numbers gives

$$\lim_{n \rightarrow \infty} \frac{1}{\tilde{\nu}(P_0|G \wedge B_n)} \sum_{j \in \mathbb{N}_0: K_j \subset B_n} \chi_{A_j}(\bar{\omega}) = \mathbb{P}_G(A_0) > 0 \quad (6.10)$$

for  $\mathbb{P}_G$ -almost all  $\bar{\omega} \in \Omega_G$ .

We conclude from the Ergodic Theorem 4.5 for uniquely ergodic systems that

$$\int_I dN = \lim_{n \rightarrow \infty} \frac{1}{\text{vol}(B_n)} \sum_{v \in \mathcal{V}_G \cap B_n} \langle \delta_v, \chi_I(H_{G(\bar{\omega})}) \delta_v \rangle, \quad (6.11)$$

for the given  $G \in X_G$  (for which (v) holds), and for all  $\bar{\omega}$  in some measurable set  $\bar{\Omega}_G \subset \Omega_G$  of full  $\mathbb{P}_G$ -measure. We choose  $\bar{\Omega}_G$  such that (6.10) holds for all  $\bar{\omega} \in \bar{\Omega}_G$ , too. Then, we rewrite

$$\begin{aligned} \sum_{v \in \mathcal{V}_G \cap B_n} \langle \delta_v, \chi_I(H_{G(\bar{\omega})}) \delta_v \rangle &= \text{tr}_{\ell^2(\mathcal{V}_G)} \left\{ \chi_{B_n} \chi_I(H_{G(\bar{\omega})}) \chi_{B_n} \right\} \\ &\geq \sum_{j \in \mathbb{N}_0: K_j \subset B_n} \frac{1}{\|\varphi_j\|^2} \langle \varphi_j, \chi_I(H_{G(\bar{\omega})}) \varphi_j \rangle \\ &\geq \sum_{j \in \mathbb{N}_0: K_j \subset B_n} \frac{\chi_{A_j}(\bar{\omega})}{\|\varphi_j\|^2} \langle \varphi_j, \chi_I(H_{G(\bar{\omega})}) \varphi_j \rangle \end{aligned} \quad (6.12)$$

and observe

$$\begin{aligned} \langle \varphi_j, \chi_I(H_{G(\bar{\omega})}) \varphi_j \rangle &= \|\varphi_j\|^2 - \|\chi_{\mathbb{R} \setminus I}(H_{G(\bar{\omega})}) \varphi_j\|^2 \\ &\geq \|\varphi_j\|^2 - \varepsilon^{-2} \int_{\mathbb{R} \setminus I} d\zeta_{G(\bar{\omega}),j}(E') (E' - E)^2 \\ &\geq \|\varphi_j\|^2 - \varepsilon^{-2} \|(H_{G(\bar{\omega})} - E) \varphi_j\|^2, \end{aligned} \quad (6.13)$$

where  $\zeta_{G(\bar{\omega}),j} := \langle \varphi_j, \chi_{\bullet}(H_{G(\bar{\omega})}) \varphi_j \rangle$  is the projection-valued spectral measure for  $H_{G(\bar{\omega})}$  and the vector  $\varphi_j$ . For  $\bar{\omega} \in A_j$ , we have  $-a_j + (G\bar{\omega} \wedge K_j) = G^\omega \wedge K_0$ . Thus, covariance of  $\hat{H}$  and (6.7) imply

$$\|(H_{G(\bar{\omega})} - E) \varphi_j\| = \|(H_{G(\omega)} - E) \varphi\| < \delta \|\varphi\| = \delta \|\varphi_j\|. \quad (6.14)$$

Combining (6.11) – (6.14), we get

$$\begin{aligned} \int_I dN &\geq (1 - \delta^2/\varepsilon^2) \liminf_{n \rightarrow \infty} \left\{ \frac{\tilde{\nu}(P_0|G \wedge B_n)}{\text{vol}(B_n)} \sum_{j \in \mathbb{N}_0: K_j \subset B_n} \frac{\chi_{A_j}(\bar{\omega})}{\tilde{\nu}(P_0|G \wedge B_n)} \right\} \\ &\geq (1 - \delta^2/\varepsilon^2) \gamma \mathbb{P}_G(A_0) > 0. \end{aligned} \quad (6.15)$$

The last inequality uses the positive lower frequency condition (6.8) and the strong law of large numbers (6.10). This completes the proof.  $\square$

The next two lemmas provide the proof of Theorem 4.7. They are consequences of the previous lemma.

**Lemma 6.4.** *Let  $\hat{H}$  be a  $\hat{\mu}$ -ergodic, self-adjoint operator of finite range and let  $N$  be its integrated density of states. Then there exists a Borel set  $\bar{X} \subseteq X_G$  with  $\mu(\bar{X}) = 1$  such that*

$$\text{supp}(dN) \subseteq \text{spec}_{\text{ess}}(H_{G(\omega)}) \quad (6.16)$$

for all  $G \in \overline{X}$  and  $\mathbb{P}_G$ -almost all  $\omega \in \Omega_G$ . If  $X_G$  is even uniquely ergodic, then the statement holds with  $\overline{X} = X_G$ .

*Proof.* By taking countable intersections of sets of full measure, we infer from the implication (i)  $\Rightarrow$  (ii) in Lemma 6.3, that there exists a Borel set  $\overline{X} \subseteq X_G$  with  $\mu(\overline{X}) = 1$  ( $\overline{X} = X_G$ , if  $X_G$  is uniquely ergodic), such that for all  $G \in \overline{X}$  there exists  $\overline{\Omega}_G \subset \Omega_G$  measurable with  $\mathbb{P}_G(\overline{\Omega}_G) = 1$ , such that for all  $\omega \in \overline{\Omega}_G$  and all  $\lambda, \lambda' \in \mathbb{Q}$  with  $\lambda < \lambda'$  we have

$$\int_{] \lambda, \lambda' [} dN > 0 \implies t_{] \lambda, \lambda' [}(G^{(\omega)}) = \infty. \quad (6.17)$$

Thus, we conclude from the characterisation (4.5) that

$$\text{supp}(dN) \subseteq \{E \in \mathbb{R} : t_{] \lambda, \lambda' [}(G^{(\omega)}) = \infty \text{ for all } \lambda, \lambda' \in \mathbb{Q} \text{ with } \lambda < E < \lambda'\} \quad (6.18)$$

for all  $G \in \overline{X}$  and all  $\omega \in \overline{\Omega}_G$ . But this is the claim.  $\square$

**Lemma 6.5.** *Let  $\widehat{H}$  be a  $\widehat{\mu}$ -ergodic, self-adjoint operator of finite range, and let  $N$  be its integrated density of states. Then there exists a Borel set  $\overline{X} \subseteq X_G$  with  $\mu(\overline{X}) = 1$ , such that*

$$\text{spec}(H_{G^{(\omega)}}) \subseteq \text{supp}(dN) \quad (6.19)$$

for all  $G \in \overline{X}$  and  $\mathbb{P}_G$ -almost all  $\omega \in \Omega_G$ . If  $X_G$  is even uniquely ergodic and if the positive lower frequency condition holds, then the statement holds with  $\overline{X} = X_G$  and for all  $\omega \in \Omega_G$ .

*Proof.* Recall that

$$\text{spec}(H_{G^{(\omega)}}) = \{E \in \mathbb{R} : t_{] \lambda, \lambda' [}(G^{(\omega)}) > 0 \text{ for all } \lambda, \lambda' \in \mathbb{Q} \text{ with } \lambda < E < \lambda'\}. \quad (6.20)$$

For uniquely ergodic systems that obey the positive lower frequency condition, we deduce the desired inclusion (6.19) – valid for all  $G \in X_G$  and all  $\omega \in \Omega_G$  – directly from the implication (v)  $\Rightarrow$  (i) in Lemma 6.3.

In the general case, we argue that there exists a Borel set  $\widehat{A} \subseteq \widehat{X}_G$  with  $\widehat{\mu}(\widehat{A}) = 1$ , such that for all  $\lambda, \lambda' \in \mathbb{Q}$  with  $\lambda < \lambda'$  the implication

$$t_{] \lambda, \lambda' [}(G^{(\omega)}) > 0 \text{ for some } G^{(\omega)} \in \widehat{A} \implies t_{] \lambda, \lambda' [}(G^{(\omega)}) > 0 \text{ for all } G^{(\omega)} \in \widehat{A}$$

holds, confer Remark 6.2 (i). Hence, for  $G^{(\omega)} \in \widehat{A}$ , we deduce the assertion from (6.20) and the implication (ii)  $\Rightarrow$  (i) in Lemma 6.3.  $\square$

## 7. PERCOLATION ESTIMATES

In this final section, we establish the percolation estimates that guarantee exponential decay of the cluster-size distribution on rather general graphs. Corollary 7.5 provides a rough criterion for this. It is used in the proof of Theorem 5.2 to ensure the applicability of Lemma 5.4. For this reason, the results of this section must be valid without any assumptions on the automorphism group of the graph. So far, this has prevented us from extending the results of this section to higher values of the percolation probability up

to the critical value. This is a challenging open problem, see also the discussion in [Ho3]. Assuming quasi-transitivity, stronger results were obtained recently in [AV2].

First we give a simple, but crude lower bound for the critical probability of Bernoulli bond percolation on an infinite connected graph  $G = (\mathcal{V}, \mathcal{E})$  with bounded degree sequence. Let  $\theta_{G,v}(p) := \mathbb{P}_G^p(|C_v| = \infty)$  denote the probability that the open cluster containing  $v \in \mathcal{V}$  is infinite. The critical probability  $p_c(G) := \sup\{p \in [0, 1] : \theta_{G,v}(p) = 0\}$  is independent of  $v \in \mathcal{V}$ , as follows from the FKG inequality, see [G, Thm. 2.8] or [Ho3] (recall that we consider only graphs with countable vertex sets). By standard reasoning [G, Thm 1.10], we have the following elementary lower estimate for  $p_c(G)$ .

**Lemma 7.1.** *Let  $G = (\mathcal{V}, \mathcal{E})$  be an infinite connected graph. If  $G$  has maximal vertex degree  $d_{\max} := \sup_{v \in \mathcal{V}} d_G(v) \in \mathbb{N} \setminus \{1\}$ , then*

$$p_c(G) \geq \frac{1}{d_{\max} - 1}. \quad (7.1)$$

*Proof.* Let  $\sigma_v(n)$  denote the number of  $n$ -step self-avoiding walks on  $G$ , starting from  $v \in V$ . Since a self-avoiding walk must not return to its previous position when performing a single step, we obtain  $\sigma_v(n) \leq d_{\max}(d_{\max} - 1)^{n-1}$ . Let  $W_v^{(\omega)}(n)$  denote the number of such walks in the percolation subgraph  $G^{(\omega)}$  of  $G$ . Since every such walk is open with probability  $p^n$  in any percolation subgraph, we get for its expectation  $\int_{\Omega_G} \mathbb{P}_G^p(d\omega) W_v^{(\omega)}(n) = p^n \sigma_v(n)$ . Note that, if the vertex  $v$  belongs to an infinite cluster, there are open self-avoiding walks of arbitrary length emanating from  $v$ . Thus, we have for all  $n \in \mathbb{N}$  the estimate

$$\begin{aligned} \theta_{G,v}(p) &\leq \mathbb{P}_G^p\{\omega \in \Omega_G : W_v^{(\omega)}(n) \geq 1\} \leq \int_{\Omega_G} \mathbb{P}_G^p(d\omega) W_v^{(\omega)}(n) \\ &= p^n \sigma_v(n) \leq \frac{d_{\max}}{d_{\max} - 1} [p(d_{\max} - 1)]^n. \end{aligned} \quad (7.2)$$

This implies  $\theta_{G,v}(p) = 0$ , if  $p(d_{\max} - 1) < 1$ , and we obtain the assertion of the lemma.  $\square$

The behaviour in the subcritical phase  $p < p_c(G)$  can be inferred from asymptotic properties of the event

$$A_v(n) := \{\omega \in \Omega_G : v \xleftrightarrow{G^{(\omega)}} B_n(v)^c\} \quad (7.3)$$

that there exists a path from  $v \in \mathcal{V}$  to the complement of the ball of radius  $n$  around  $v$ . For fixed  $p$ , denote by  $g_{G,v}^p(n) := \mathbb{P}_G^p(A_v(n))$  the probability of the event  $A_v(n)$ . The following lemma states that  $g_{G,v}^p(n)$  decays exponentially in  $n$ , if the percolation probability  $p$  is small enough. Its proof is analogous to that of the previous lemma.

**Lemma 7.2.** *Let a graph  $G = (\mathcal{V}, \mathcal{E})$  be given. Assume that  $G$  has maximal vertex degree  $d_{\max} \in \mathbb{N} \setminus \{1\}$  and maximal edge length  $l_{\max} := \sup\{|u - v| : \{u, v\} \in \mathcal{E}\} < \infty$ . Then, for every  $p \in [0, 1]$  there exists a real number  $\psi(p)$ ,*

such that the probability  $g_{G,v}^p(n)$  of the event  $A_v(n)$  satisfies

$$g_{G,v}^p(n) \leq 2e^{-n\psi(p)} \quad (7.4)$$

for all  $n \in \mathbb{N}$ , uniformly in  $v \in \mathcal{V}$ . A possible choice of  $\psi(p)$  is, for  $0 < p \leq 1$ ,

$$\psi(p) = \frac{1}{l_{\max}} \ln \left( \frac{1}{p(d_{\max} - 1)} \right). \quad (7.5)$$

In particular,  $g_{G,v}^p(n)$  decays exponentially if  $p < 1/(d_{\max} - 1)$ .

*Proof.* A path with initial vertex  $v$ , which enters the complement of  $B_n(v)$ , contains at least  $\tilde{n} := \lceil n/l_{\max} \rceil$  bonds, where  $\lceil x \rceil$  is the smallest integer  $\geq x$ . With the notation in the proof of Lemma 7.1, the assertion now follows by noting that

$$g_{G,v}^p(n) \leq \mathbb{P}_G^p \{ \omega \in \Omega_G : W_v^{(\omega)}(\tilde{n}) \geq 1 \} \leq \frac{d_{\max}}{d_{\max} - 1} [p(d_{\max} - 1)]^{\tilde{n}}. \quad (7.6)$$

□

For a graph  $G = (\mathcal{V}, \mathcal{E})$  with uniformly discrete vertex set, exponential decay of  $g_{G,v}^p(n)$  implies that the mean cluster size  $\chi_{G,v}(p) := \mathbb{E}_G^p\{|C_v|\}$  is finite, as follows from the argument in [G, p. 89]. In fact, this argument yields an estimate which is uniform in  $v \in \mathcal{V}$ . We state the result as

**Lemma 7.3.** *Let  $G = (\mathcal{V}, \mathcal{E})$  be a graph with uniformly discrete vertex set of radius  $r > 0$ . Assume that  $G$  has maximal vertex degree  $d_{\max} \in \mathbb{N} \setminus \{1\}$  and maximal edge length  $l_{\max} < \infty$ . Then, for every  $p \in [0, \frac{1}{d_{\max}-1}[$ , there exists a constant  $\chi(p) \in ]0, \infty[$ , which depends only on  $r$ ,  $d_{\max}$  and  $l_{\max}$  otherwise, such that the mean cluster size  $\chi_{G,v}(p) = \mathbb{E}_G^p\{|C_v|\}$  satisfies*

$$\chi_{G,v}(p) \leq \chi(p) < \infty, \quad (7.7)$$

uniformly in  $v \in \mathcal{V}$ .

□

Lemma 7.2 and Lemma 7.3 can be used to prove exponential decay of the cluster size distribution.

**Theorem 7.4.** *Let  $G = (\mathcal{V}, \mathcal{E})$  be a graph with uniformly discrete vertex set of radius  $r > 0$ . Assume that  $G$  has maximal vertex degree  $d_{\max} \in \mathbb{N} \setminus \{1\}$  and maximal edge length  $l_{\max} < \infty$ . Then, for every  $p \in [0, \frac{1}{d_{\max}-1}[$ , there exists a constant  $\lambda(p) \in ]0, \infty[$ , which depends only on  $r$ ,  $d_{\max}$  and  $l_{\max}$  otherwise, such that*

$$\mathbb{P}_G^p(|C_v| \geq n) \leq 2e^{-n\lambda(p)} \quad (7.8)$$

for all  $n \in \mathbb{N}$ , uniformly in  $v \in \mathcal{V}$ .

*Sketch of the proof.* Proceed along the lines of [G, Thm. 6.75]. In the estimates, replace  $\chi_{G,v}(p)$  by its uniform bound  $\chi(p)$ . The decay rate thus obtained is  $\lambda(p) = [2\chi(p)^2]^{-1}$ . □

The conclusion in the above theorem still holds if, instead of a single graph  $G$ , we consider a set of graphs  $\mathcal{G}$  with the above properties and the associated dynamical system  $X_{\mathcal{G}}$ . This setup is used in Section 5.



**Corollary 7.5.** *Let  $\mathcal{G}$  be a set of graphs, whose vertex sets are uniformly discrete of radius  $r > 0$ , which have maximum vertex degree  $d_{\max} \in \mathbb{N} \setminus \{1\}$  and maximal edge length  $l_{\max} < \infty$ . Then, for every  $p \in [0, \frac{1}{d_{\max}-1}[$  there exists a constant  $\lambda(p) \in ]0, \infty[$  such that*

$$\mathbb{P}_G^p(|C_v| \geq n) \leq 2e^{-n\lambda(p)} \quad (7.9)$$

for all  $n \in \mathbb{N}$ , uniformly in  $G \in X_{\mathcal{G}}$  and in  $v \in \mathcal{V}_G$ .  $\square$

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